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*University of Mining and Metallurgy, Faculty of Materials Science and Ceramics, Cracow

**Jagiellonian University, Institute of Mathematics, Cracow

The Conservation of Momentum and Energy in Open Systems, Diffusional Approach to the Mass Transport in Solids

By

M. Danielewski* and W. Krzywański**

The Navier-Stokes equations do not describe the mixing of mass in a continuous medium. In this work we analyze the motion of mixture in which mass flows from the region of higher mass density into the lower one. In such a case the mass is transported as a result of pressure, viscosity, external forces and diffusion. We consider a motion of continuous medium in which the diffusional flux of mass occurs. The momentum balance of mixture in which diffusion is considered causes the new term to appear in the Navier-Stokes equation. Apart from terms responsible for pressure, viscosity and external forces there is the diffusional one. We analyze whether the internal energy of mixture depends on diffusion. The conclusion is that diffusional flux does not change the internal energy of mixture. We define the specific initial conditions for a pure diffusion and elucidate that it is the diffusion, described by the well-known equation of diffusion.

1) Al. Mickiewicza 30, PL 30-059, Poland.

2) ul. Reymonta 4, PL 30-059, Poland.
1. Introduction

The mass transport process in multi-component media (e.g. in either amorphous or crystalline solids) can consist of a complex set of interactions between the diffusional fluxes of each element. When the mixture (e.g. solid solution) forming components have different partial molar volumes and/or mobilities, their simultaneous diffusion can result in an imbalance in the "volume" transport, can generate drift velocity [1]. The interdiffusion in solids produces the convective mass transport (e.g. the Kirkendall effect [2]). The imbalance in the "volume" transport generates the elastic strain and internal stress.

In the fundamental analysis of convective transport during interdiffusion by Darken [1], it is assumed that the general mass flux is a sum of diffusional and drift (convective) fluxes only. Namely he postulated that the velocity of the $i$-th component in an unidimensional mixture can be expressed by

$$v_i := - \Theta_i \frac{\partial c_i}{\partial x} + \Theta_i v,$$  \hspace{1cm} (1)

where $v_i$ denotes the velocity of the $i$-th component, $\Theta_i$ and $\Theta_i$ are its diffusivity and density, respectively. In general, the diffusivity (Darken's intrinsic diffusion coefficient) can depend on mixture composition. In the Darken model of interdiffusion the contributions of stress and defects chemical potential to the diffusion potentials are neglected [3].

The alternative approaches to phenomenological description of the interdiffusion were presented by Stephenson [4]. Their essence is the interaction between stress and diffusion [5]. The models are restricted to the uni-dimensional binary systems and quasi-stationary processes only. The Stevenson approach base on the Darken's concept of the separation of diffusional and drift (convective) flows and the variable intrinsic diffusion coefficient [1]. Moreover it is restricted to the semi-infinite
(unbouded) system.

The separation of diffusion and translation is also a basic concept in the recent approach to interdiffusion in a multicomponent, unidimensional bounded system [6]. It has been shown that the equations of mass conservation (continuity equations), the expressions describing the fluxes (translation flux and diffusional flux) and momentum conservation equation (equation of motion) allow a complete quantitative description of diffusional transport process (in a mixture showing constant concentration) to be formulated. The equations describing the interdiffusion process (mixing) in the general case where the components diffusivities vary with composition has been derived. When certain regularity assumptions (concerning initial data) and, a quantitative condition (concerning the diffusion coefficients - providing a parabolic type of the final equation) are fulfilled, then there exists the unique solution of such interdiffusion problem.

The all above mentioned models of transport base on the Darken's concept of the common medium translation velocity (i.e. the drift flow or convective flow). Consequently (unavoidably when the three-dimensional problems are analysed), this additional variable requires the formula which will balance number of unknowns and equations (a mathematical formula of momentum conservation is required).

The starting equation of the dynamics of solids is the general integral formula of momentum conservation

$$\frac{d}{dt} \int_{\Omega(t)} \rho u \, dx = - \int_{\partial\Omega(t)} \rho u \nu \, dS + \int_{\Omega(t)} \rho f \, dx + \int_{\partial\Omega(t)} t(n) \, dS ,$$

(2)

where \(t(n)\) denotes the total strain tensor (other notations will be introduced in the following section).

The starting integral formula of the equation of momentum conservation in dynamics of fluids has a form [7]
\[
\frac{d}{dt} \int_{\Omega(t)} \rho \mathbf{u} \, d\mathbf{x} = - \int_{\partial \Omega(t)} \mathbf{n} \cdot \mathbf{u} \, d\mathbf{S} + \int_{\Omega(t)} \mathbf{f} \, d\mathbf{x} + \int_{\partial \Omega(t)} \mathbf{p} \, d\mathbf{S},
\]

where \( \mathbf{p} \) denotes the pressure tensor.

The development of the differential formula of law of momentum conservation in solids base on the careful subdivision of the total strain into elastic, plastic and stress-free strains while in case of fluids the properties of pressure tensor are of interest.

In the present treatment we base on the phenomenology of Navier-Stokes equations. In order to simplify the all derived formula we restrict our analysis to the amorphous solids (in the general case of crystalline solids the diffusivities and viscosity coefficients become a second order tensors).

To understand the heuristics of the postulated diffusional term in equations describing transport in continuous medium, let us consider a difference between the kinematic and dynamic model of interdiffusion.

In Fig. 1 the interdiffusion of two pure metals is shown. The diffusional couple is placed on a horizontal surface and has a zero-friction roller system. At the beginning metal A and B occupy the same volume (Fig. 1a). For a positive time the diffusion process results in the change of the density profile, Fig. 1b. After a sufficiently long time the homogeneous alloy will be formed. Fig. 1c. Because there is no net external force acting on the diffusional couple, it follows that as a result of changed mass distribution in the couple (affected by the interdiffusion process), its position has to be changed in such a way that the position of mass center is constant (\( x_m = \text{const.} \)).

Thus by Newton's first law of motion a **non-zero translation of the container** has to be observed and this effect is caused by the interdiffusion process.

The analogous example could be given in the case of diffusion of hydrogen in palladium and several others. All of them exemplify the conclusion (starting message of our work) that the diffusional transport of mass can affect the local translation velocity of all continuous media. Such
effects are ignored in the kinematic models of interdiffusion.

2. Theory

Notation. Let us consider a motion of medium (r-component mixture). Let $v_i: [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the velocity of the $i$-th component and $\rho_i: [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be its density of mass. The sign $X \rightarrow Y$ means that the domain of a function is a subset of $X$. Let $\Omega$ denote any subregion of the moving medium and $\Omega(t)$ to be the set of the points of fluid that belonged to $\Omega$ at the moment $t = 0$. Moreover we define: the density of the medium

$$\rho := \sum_{i=1}^{r} \rho_i,$$

(4)

the flux (total flux) of the medium

$$\rho u := \sum_{i=1}^{r} \rho_i v_i,$$

(5)

where $u$ denotes the velocity of the local centre of mass and, the diffusional flux (total diffusional flux in the medium)

$$\rho v_d := \sum_{i=1}^{r} \rho_i v_{d,i} = \sum_{i=1}^{r} j_{d,i},$$

(6)
where the diffusional flux of the \( i \)-th component, \( j_{d,i} \), can be expressed by any appropriate formula.

The basic theorem that we apply in our considerations is due to Liouville.

**THEOREM.** If \( \varphi : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R} \) or \( \varphi : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3 \) is a sufficiently smooth function defined on domain \( \mathcal{U} \) then

\[
\frac{d}{dt} \int_{\mathcal{Q}(t)} \varphi \, dx = \int_{\mathcal{Q}(t)} \left( \left( \frac{\partial \varphi}{\partial t} + \text{div}(\varphi \mathbf{u}) \right) \varphi + \mathbf{u} \left( \frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{w} \right) \right) \, dx.
\]

The proof of the above theorem can be found elsewhere [8]. We will derive the well known equation of mass conservation in order to demonstrate the routine of using the Liouville theorem.

**Conservation of Mass.** The mass \( m \) contained by \( \mathcal{Q}(t) \) at the moment \( t \) is

\[
m = \int_{\mathcal{Q}(t)} \varphi \, dx.
\]

The principle of conservation of mass states that the mass of medium in \( \mathcal{Q}(t) \) does not change. Thus

\[
\int_{\mathcal{Q}(t)} \varphi \, dx = \text{const}.
\]

By Liouville theorem \( \varphi \equiv 1 \)

\[
0 = \frac{d}{dt} \int_{\mathcal{Q}(t)} \varphi \, dx = \int_{\mathcal{Q}(t)} \left( \frac{\partial \varphi}{\partial t} + \text{div}(\varphi \mathbf{u}) \right) \, dx.
\]

Since the subregion \( \Omega \) was chosen arbitrarily, then
\[
\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0. \tag{7}
\]

The equation (7) is known as the continuity equation.

**Balance of Momentum.** The momentum of medium in an evolving subregion \( \Omega(t) \) is

\[
\text{momentum of fluid in } \Omega(t) = \int_{\Omega(t)} \rho \mathbf{u} \, dx.
\]

By Newton's second law, the total force, \( F_{\text{total}} \), acting on mass in \( \Omega(t) \) is given by:

\[
F_{\text{total}} = \frac{d}{dt} \int_{\Omega(t)} \rho \mathbf{u} \, dx.
\]

We assume that there are the following forces acting on the mass in \( \Omega(t) \):

1° The force of stress, \( F_p \), exerted across the surface \( \partial \Omega(t) \)

\[
F_p = - \int_{\partial \Omega(t)} p \mathbf{n} \, dS,
\]

where a function \( p : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R} \) is called the pressure (pressure tensor in general), \( n \) is a versor normal to \( \partial \Omega(t) \) pointing outward of \( \Omega(t) \) and \( dS \) is the surface element.

2° The viscosity force, \( F_\eta \), acting on the surface \( \partial \Omega(t) \)

\[
F_\eta = \int_{\partial \Omega(t)} \sigma(\mathbf{n}) \, dS,
\]
where $\sigma: [0, \infty) \times \mathbb{R}^3 \to L(\mathbb{R}^3, \mathbb{R}^3)$ is the viscosity tensor (corresponds to Landau notation [9]) and $L(\mathbb{R}^3, \mathbb{R}^3)$ is the space of linear mappings from $\mathbb{R}^3$ to $\mathbb{R}^3$.

$F_{\text{ext}}$, acting on the mass in the volume $\Omega(t)$

$$F_{\text{ext}} = \int_{\Omega(t)} f \, dx ,$$

where $f: [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ is a density of $F_{\text{ext}}$.

Our goal is to elucidate the consequences of diffusional mass transport. Thus, apart of the forces mentioned above, we will add a new one connected with diffusion (e.g. interdiffusion).

If $v_d$ denotes the velocity of mass caused by diffusion, eq.(6), then the momentum of medium in $\Omega(t)$ due to mass transport with velocity $v_d$ (i.e. momentum of mixture due to diffusion) is

$$\text{momentum of medium in } \Omega(t) \text{ caused by diffusion} = \int_{\Omega(t)} \rho \, v_d \, dx .$$

Then by the Newton's second law:

$F_d$ equals

$$F_d = \frac{d}{dt} \int_{\Omega(t)} \rho \, v_d \, dx .$$

The total force acting on mass in $\Omega(t)$ is the sum of forces listed in $\mathbb{R}^3 - \mathbb{R}^3$. Consequently:

$$F_{\text{total}} = F_\eta + F_p + F_{\text{ext}} + F_d .$$
Applying the formulae from \( 1^o - n^o \) results in:

\[
\frac{d}{dt} \int_{\Omega(t)} \rho u \, dx = - \int_{\partial \Omega(t)} \rho n \, dS + \int_{\Omega(t)} \sigma(n) \, dS + \int_{\Omega(t)} f \, dx + \frac{d}{dt} \int_{\Omega(t)} \rho v_d \, dx .
\]  

(8)

With the use of Liouville theorem, the integral form of the balance of momentum can be converted to the differential one. By this theorem first and last terms in Eq.(8) become:

\[
\frac{d}{dt} \int_{\Omega(t)} \rho u \, dx = \int_{\Omega(t)} \left( \left( \frac{\partial \rho}{\partial t} + \text{div}(\rho u) \right) u + \rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) \right) \, dx ,
\]

\[
\frac{d}{dt} \int_{\Omega(t)} \rho v_d \, dx = \int_{\Omega(t)} \left( \left( \frac{\partial \rho}{\partial t} + \text{div}(\rho u) \right) v_d + \rho \left( \frac{\partial v_d}{\partial t} + (u \cdot \nabla)v_d \right) \right) \, dx .
\]

Taking use of the equation of continuity, Eq.(7), the above formulae simplifies to the form:

\[
\frac{d}{dt} \int_{\Omega(t)} \rho u \, dx = \int_{\Omega(t)} \rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) \, dx ,
\]  

(9)

\[
\frac{d}{dt} \int_{\Omega(t)} \rho v_d \, dx = \int_{\Omega(t)} \rho \left( \frac{\partial v_d}{\partial t} + (u \cdot \nabla)v_d \right) \, dx .
\]  

(10)

Transforming the first and second terms (on the right hand side of Eq.(8)) from surface integrals to volume integrals by Gauss' theorem we obtain:

\[
\int_{\partial \Omega(t)} \rho n \, dS = \int_{\Omega(t)} \nabla \rho \, dx ,
\]  

(11)
\[ \int_{\Omega(t)} \sigma(n) \, dS = \int_{\Omega(t)} \sum_{j=1}^{3} \left( \frac{\partial \sigma_{1j}}{\partial x_j}, \frac{\partial \sigma_{2j}}{\partial x_j}, \frac{\partial \sigma_{3j}}{\partial x_j} \right) \, dx, \]  

where \( \sigma = (\sigma^1, \sigma^2, \sigma^3) \), \( \sigma^j(e_j) = \sigma_{ij} \) and \( e_1, e_2, e_3 \) is the standard basis in \( \mathbb{R}^3 \). Thus substituting Eqs. (9) - (11) into Eq. (8) one gets

\[ \int_{\Omega(t)} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \, dx = - \int_{\Omega(t)} \nabla p \, dx + \int_{\Omega(t)} \sum_{j=1}^{3} \left( \frac{\partial \sigma_{1j}}{\partial x_j}, \frac{\partial \sigma_{2j}}{\partial x_j}, \frac{\partial \sigma_{3j}}{\partial x_j} \right) \, dx + \int_{\Omega(t)} f \, dx + \int_{\Omega(t)} \rho \left( \frac{\partial v_d}{\partial t} + (\mathbf{u} \cdot \nabla) v_d \right) \, dx. \]  

Since the subregion \( \Omega(t) \) was chosen arbitrarily, we can omit the integral and write Eq. (13) in a differential form:

\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = - \nabla p + \sum_{j=1}^{3} \left( \frac{\partial \sigma_{1j}}{\partial x_j}, \frac{\partial \sigma_{2j}}{\partial x_j}, \frac{\partial \sigma_{3j}}{\partial x_j} \right) + f + \rho \left( \frac{\partial v_d}{\partial t} + (\mathbf{u} \cdot \nabla) v_d \right). \]

If we denote \( \left. \frac{D}{Dt} \right|_u = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \) and \( \nabla \sigma = \sum_{j=1}^{3} \left( \frac{\partial \sigma_{1j}}{\partial x_j}, \frac{\partial \sigma_{2j}}{\partial x_j}, \frac{\partial \sigma_{3j}}{\partial x_j} \right) \), then the above equation reduces to

\[ \rho \left. \frac{D \mathbf{u}}{Dt} \right|_u = - \nabla p + \text{Div} \sigma + f + \rho \left. \frac{D v_d}{Dt} \right|_u. \]
The equation (14) we shall call the equation of motion. An attempt to incorporate the momentum flux due to the diffusion of mass (into the general formula of equation of motion) has been already published [10].

The viscosity tensor $\sigma$ is assumed to have the form:

$$\sigma_{ij} = \eta \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \operatorname{div} \mathbf{u} \right) + \zeta \delta_{ij} \operatorname{div} \mathbf{u},$$

(15)

where the constants $\eta$ and $\zeta$, are the first and second coefficients of mixture viscosity [9].

The quantity $\rho \mathbf{v}_d$ is called the diffusional flux, eq.(6). The partial diffusional flow, the diffusional flow of the $i$-th component, can be expressed by any appropriate formula, e.g. by Fick's law:

$$\rho \mathbf{v}_{d,i} = - \Theta_i(\rho, \ldots, \rho_i) \nabla \rho_i,$$

(16)

where $\Theta_i(\rho, \ldots, \rho_i)$ denotes the diffusion coefficient, which in general depends on all variables.

**Balance of Energy.** We shall show that the diffusional forces do not change the internal energy of the medium. We assume that the total energy of moving medium in $\Omega(t)$ can be written as:

$$\begin{align*}
\mathbf{e} &= \mathbf{e}_k + \mathbf{e}_i, \\
\text{[total energy of mass in } \Omega(t)] &= \text{[kinetic energy]} + \text{[internal energy]}
\end{align*}$$

(17)

The kinetic energy, $e_k$, contained in the moving subregion $\Omega(t)$ is

$$e_k = \int_{\Omega(t)} \frac{1}{2} \rho \mathbf{u}^2 \, dx.$$  

(18)
The internal energy of mass in $\Omega(t)$ is represented by its density per unit of mass, $\varepsilon$, in the following way

$$ e_1 = \int_{\Omega(t)} \rho \varepsilon \, dx . \quad (19) $$

By the First Law of Thermodynamics the change of the total energy of portion of medium in $\Omega(t)$ equals the sum of work which is done on it and net heat transported into $\Omega(t)$. Thus from Eqs.(18) and (19):

$$ \left[ \text{rate of change of total energy in } \Omega(t) \right] = \frac{d}{dt} \int_{\Omega(t)} \rho \left( \frac{1}{2} u^2 + \varepsilon \right) \, dx . \quad (20) $$

The rate of work done on medium in $\Omega(t)$ is [8]:

$$ \left[ \text{rate of work done on mass in } \Omega(t) \right] = - \int_{\partial \Omega(t)} p \cdot n \, dS + \int_{\Omega(t)} u \cdot \sigma(n) \, dS + \int_{\Omega(t)} u \cdot f \, dx + \int_{\Omega(t)} \rho u \cdot \frac{Dv}{Dt} \, dx . $$

The rate of change of internal energy due to the thermal conductivity (i.e. heat flux, $j_q$) is represented by:

$$ \left[ \text{rate of change of internal energy in } \Omega(t) \text{ as a result of heat flux} \right] = - \int_{\partial \Omega(t)} j_q \cdot n \, dS . $$

Upon substituting Eqs.(20) - (22), the First Law of Thermodynamics (applied to the portion of mixture in $\Omega(t)$) can be written in the form:
Applying Liouville theorem and the continuity equation, the term on the left hand side of Eq.(23) becomes:

\[
\frac{d}{dt} \int_{\Omega(t)} \rho \left( \frac{1}{2} u^2 + e \right) \, dx = \int_{\Omega(t)} \rho u \cdot \partial_t \left( u + (u \cdot \nabla) u \right) + \frac{\partial e}{\partial t} + (u \cdot \nabla) e \, dx.
\]  

(23)

Transforming the surface integrals on the r.h.s of Eq.(23) to the volume ones, we get:

\[-\int_{\partial \Omega(t)} p u \cdot n \, d\mathcal{S} = \int_{\Omega(t)} \left( -u \cdot \nabla p - p \, \text{div} u \right) \, dx,\]

(25)

\[
\int_{\partial \Omega(t)} u \cdot \sigma(n) \, d\mathcal{S} = \int_{\Omega(t)} \left( \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j} \sigma_{ij} + u \cdot \text{Dive} \sigma \right) \, dx,
\]

(26)

\[
\int_{\partial \Omega(t)} j_q \cdot n \, d\mathcal{S} = \int_{\Omega(t)} \text{div} j_q \, dx.
\]

(27)

Combining Eqs.(23) - (27):

\[
\int_{\Omega(t)} \rho \left( u \cdot \left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) + \frac{\partial e}{\partial t} + (u \cdot \nabla) e \right) \, dx = \int_{\Omega(t)} \left( -u \cdot \nabla p - p \, \text{div} u \right) \, dx + \int_{\Omega(t)} \rho u \cdot \frac{D v}{Dt} \, dx + \int_{\Omega(t)} u \cdot f \, dx - \int_{\Omega(t)} \text{div} j_q \, dx.
\]  

(28)
Since the subregion $\Omega(t)$ was chosen arbitrarily, the integrals in Eq.(28) can be omitted:

$$\rho \left( u \cdot \left( \frac{\partial u}{\partial t} + \left( u \cdot \nabla \right) u \right) + \frac{\partial \varepsilon}{\partial t} + \left( u \cdot \nabla \right) \varepsilon \right) = - u \cdot \nabla p - p \text{ div } u +$$

$$+ \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \sigma_{ij} + u \cdot \text{Div } \sigma + \rho u \cdot \frac{D v_d}{Dt} \bigg|_a + u \cdot f - \text{ div } j_q .$$

(29)

Multiplying the both sides of equation of motion by $u$ we get the local conservation equation of the kinetic energy of translation:

$$\rho u \cdot \left( \frac{\partial u}{\partial t} + \left( u \cdot \nabla \right) u \right) = - u \cdot \nabla p + u \cdot \text{Div } \sigma + u \cdot f + \rho u \cdot \frac{D v_d}{Dt} \bigg|_a .$$

(30)

Consequently Eq.(29), upon substituting Eq.(30), becomes:

$$\rho \left( \frac{\partial \varepsilon}{\partial t} + \left( u \cdot \nabla \right) \varepsilon \right) = - p \text{ div } u + \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \sigma_{ij} - \text{ div } j_q ,$$

(31)

in the condensed, equivalent form it can be written as

$$\rho \frac{D \varepsilon}{Dt} \bigg|_a = - p \text{ div } u + \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \sigma_{ij} - \text{ div } j_q .$$

This is differential form of the First Law of Thermodynamics. The heat flux, $j_q$, has the form

$$j_q = - \kappa \nabla T ,$$

(32)

where $T$ is the temperature of the mixture and $\kappa$ denotes the coefficient of thermal conductivity.

It is evident that the internal energy does not depend on the diffusional flux, $\rho v_d$. Thus, the diffusion does not change the internal energy of moving fluid.

**Pure Diffusion.** We define the pure diffusion as a state of medium in which the only force acting
on every part of it is the diffusional one and, that the medium is the single-component one. The equation of motion for such a medium reduces to:

\[
\varrho \frac{D u}{Dt} \bigg|_x = \varrho \frac{D v_d}{Dt} \bigg|_x .
\]  

(33)

Using Liouville theorem one can prove [11]:

**THEOREM.** If at the moment \( t = 0 \) \( u(0, x) = v_d(0, x) \) for all \( x \in \text{domain } u(0, \cdot) \) and the vector fields \( u, v_d \) satisfy (33) then for any \( t \geq 0 \) and \( x \in \text{domain } u(t, \cdot) \):

\[ u(t, x) = v_d(t, x). \]

Thus, the medium in the state of pure diffusion moving at the moment \( t = 0 \) with the velocity \( v_d(0, \cdot) \) has the density of momentum (Eq.(16))

\[
\varrho u = -\Theta \nabla \varrho .
\]  

(34)

Since the continuity equation, Eq.(7), is valid we get

\[
\frac{\partial \varrho}{\partial t} = \text{div} (\Theta \nabla \varrho) .
\]  

(35)

If \( \Theta = \text{const.} \) then Eq.(35) can be transformed to

\[
\frac{\partial \varrho}{\partial t} = \Theta \Delta \varrho.
\]  

(36)

Hence the density of fluid in pure diffusion satisfies the well-known equation of diffusion.

3. Summary

In the paper we have analyzed a motion of continuous medium in which a transport of mass (diffusion) is present. The principle of mass conservation implies the well known continuity
equation. On the base of Newton's law we have obtained the equation of motion (local momentum conservation equation):

\[ \rho \frac{Du}{Dt} \bigg|_a = \nabla p + \text{Div} \sigma + f + \rho \frac{Du_d}{D\tau_a} \bigg|_a , \]

where \( \rho u_d \) denotes the average diffusional mass flux which can be expressed by any suitable formula, e.g. in the form of the Fickian flows.

The equation of motion has a form of Navier-Stokes equation, except the additional diffusional term

\[ f_d = \rho \frac{Du_d}{Dt} \bigg|_a . \]

We have shown that the above diffusional term (diffusional force, \( f_d \)) does not affects the internal energy of medium. Consequently the equation of internal energy conservation, eq.(31), has the usual form:

\[ \rho \frac{D\varepsilon}{Dt} \bigg|_a = -p \text{Div} \varepsilon + \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \sigma_{ij} - \text{Div} f_q . \]

Finally we have examined the motion of the single component medium as a result of diffusion only, when the pressure (\( p = \text{const.} \)), viscosity (\( \zeta = \eta = 0 \)) and external forces (\( f_d \)) are negligible. We have concluded that in such a case the medium satisfies the well-known equation of diffusion:

\[ \frac{\partial \rho}{\partial t} = \Theta \Delta \rho . \]

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References


Figure captions

Figure 1. The interdiffusion in the closed container placed on a zero-friction roller system, the net external forcing is negligible \( \sum_{k=1}^{n} F_j = 0 \) : a) the initial distribution \( (t = 0) \), b) interdiffusion "in action" - the distributions are altered \( \varrho_t = \varrho(t, x) \) and c) the final phase \( (t >> 0) \) - the homogeneous ally.